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# Temporal intermittency caused by ion-neutral drift

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**Abstract**

In a plasma formed by charged and neutral particles, the first species should evolve according to the laws of magnetohydrodynamics while the second would obey the Navier–Stokes equations. However, collisions between both species provide a friction that couples their motion and has important consequences in the long-term evolution of the plasma. It is found that there exist long periods of stability where the energy of each species as well as the magnetic energy tends to be roughly constant, punctuated by short intervals of rapid transfer of energy. Since at these events both the Lorentz force and the Hall term are very large, they could be identified as episodes of fast magnetic reconnection.

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## 1. Introduction

In several astrophysical phenomena the interplay between plasma flow and magnetic field seems to have some degree of time intermittency. The classical example is the sudden transformation from magnetic to kinetic energy present in solar flares, although this occurs in a fully ionized plasma, not in the partially ionized ones we will study. More appropriate would be the localized heating in a dense interstellar medium [1] or the formation of meteoritic chondrules in protoplanetary discs [2]. We will prove rigorously that there exists a situation where long-term evolution, if it allows any interchange between kinetic and magnetic energy, must do so in an intermittent manner. Moreover, whenever an energy transfer exists, all classical ingredients of magnetic reconnection are present: there must exist a large Lorentz force, and therefore a large mean current, and the curl of the Lorentz force (the so-called Hall term) is also large. The situation we are describing occurs when the plasma is not totally ionized, and thus part of it does not respond to the magnetic field; however, ions and neutrals collide often enough to produce a friction which makes both species to tend to drag together. Electrons and ions, however, are assumed coupled, so that we do not deal with electron magnetohydrodynamics (MHD) but rather with the conventional Navier–Stokes and MHD equations coupled by the collision term. This causes the so-called ion-neutral or ambipolar

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drift, which plays an important role e.g. in galactic dynamos. It was introduced as far back as 1956, by Mestel and Spitzer [3], and its formulation is straightforward. We will assume that both species are inviscid, which is a fair approximation in astrophysical problems, and incompressible: this certainly does not hold in galactic or stellar plasmas, but we adopt it in the interest of simplicity. Then the velocity $\mathbf{v}_n$ of the neutrals satisfies the Navier–Stokes (or rather the Euler) equation

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} = -\rho_n \mathbf{v}_n \cdot \nabla \mathbf{v}_n - \nabla P_n + \rho_n \nu_{ni} (\mathbf{v}_i - \mathbf{v}_n), \tag{1}$$

where $\rho_n$ is the (constant) density of the neutrals, $P_n$ the kinetic pressure, $\nu_{ni}$ the neutral-ion collision frequency, $\mathbf{v}_i$ the ion velocity: the last term provides the drag between both species.

As for the ion velocity, it also satisfies the Euler equation, but with the forcing provided by the Lorentz force:

$$\rho_i \frac{\partial \mathbf{v}_i}{\partial t} = -\rho_i \mathbf{v}_i \cdot \nabla \mathbf{v}_i - \nabla P_i + \rho_i \nu_{in} (\mathbf{v}_n - \mathbf{v}_i) + \mathbf{J} \times \mathbf{B}, \tag{2}$$

where $\rho_i$ is the ion density, $P_i$ the kinetic pressure, $\nu_{in}$ the ion-neutral collision frequency: one has $\rho_i \nu_{in} = \rho_n \nu_{ni} = k > 0$. In the limit $k \to 0$ we would have two uncoupled equations, a Navier–Stokes one for the neutral particles and an MHD system for the ions, and both species would follow their own path without interfering. In the limit $k \to \infty$, as we will see, either both velocities are equal or they would collapse catastrophically to zero. $\mathbf{B}$ is the magnetic field, $\mathbf{J} = \nabla \times \mathbf{B}$ the current density. There would be no mathematical problem adding viscosity and its associated diffusive terms, but we want to emphasize precisely that even in the absence of diffusion the collisions provide a sink of kinetic energy which accounts for a peculiar evolution: the presence of viscosity would only mask these characteristics.

As for the magnetic field, we could take by analogy the ideal (no resistivity) induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_i \times \mathbf{B}). \tag{3}$$

Nonetheless, in our case we must refine this equation somewhat if we wish to account for energy conversion through magnetic reconnection. It is well known that early efforts to explain fast reconnection in terms of classical resistivity were not entirely successful. The problem lies in the fact that the reconnection region is so narrow that the flow of plasma towards it gets throttled by the slow ejection [4, 5]. While the full description of what happens at the current sheets, where ions and electrons part, needs the two-fluid MHD equations, the inclusion of the Hall term $-h\nabla \times (\mathbf{J} \times \mathbf{B})$ is enough to obtain a workable model. Resistivity should also be present, even if its value is very small. Thus we take as induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (-\eta \mathbf{J} + \mathbf{v}_i \times \mathbf{B} - h\mathbf{J} \times \mathbf{B}), \tag{4}$$

where we assume that all the magnitudes, and therefore all constants, have been non-dimensionalized by dividing by some typical parameters e.g. length scale, mean magnetic field strength, Alfvén velocity, etc. An appropriate choice of these reference magnitudes is important in numerical models and when numerical data are presented, but for our qualitative purposes they are not that relevant. We will assume, however, that the magnetic Reynolds number $\eta^{-1}$ is large, i.e. that $\eta$ is much smaller than the remaining constants $k$ and $h$, $h$ being the Hall coefficient. Thus the Ohmic loss of magnetic energy works on much larger time scales than the other effects.

In weakly ionized plasmas $\rho_i \sim 0$, which means that the Lorentz force must balance the friction:

$$\mathbf{v}_i - \mathbf{v}_n = \frac{\mathbf{J} \times \mathbf{B}}{k}. \tag{5}$$

In this case, called the strong coupling approximation, numerical simulations apparently show the generation of sharp magnetic structures [6, 7], which favour the presence of magnetic reconnection. We will not, however, make this hypothesis and allow for arbitrary relative velocities of ions and neutral particles. Hence we take (1), (2) and (4) as our basic equations, and assume that solutions smooth enough exist for all time. As for the boundary conditions, the order of each equation must be taken into account in order not to overdetermine the system. Thus, if we consider that the flow does not leave the smooth domain $\Omega$, the normal component of both velocities must be zero at its boundary, and no further boundary conditions may be imposed:

$$\mathbf{v}_n \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{v}_i \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

As for the magnetic field, since it obeys the parabolic equation (4), three conditions may be imposed. A standard one is $\mathbf{B} \cdot \mathbf{n} = 0$ at $\partial\Omega$, which would occur e.g. if the field outside $\Omega$ is zero. However, in most cases this is not so; moreover, the classical picture of formation of current sheets [8] involves the motion of the foot points of magnetic field lines at the boundary, transported there by $\mathbf{v}_i$: when the field is tangential to the boundary this argument loses its meaning. What really happens, at least for insulator boundaries, is $\mathbf{J} \cdot \mathbf{n} = 0$ at $\partial\Omega$. This suggests choosing $\mathbf{B} \times \mathbf{n} = \mathbf{0}$ at the boundary as our condition for the field; we could still take a third condition, but it will be irrelevant to our proofs. Therefore we assume

$$\mathbf{v}_n \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{v}_i \cdot \mathbf{n}|_{\partial\Omega} = 0; \qquad \mathbf{B} \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}. \tag{6}$$

## 2. Energy balance

To obtain the main energy inequalities we multiply (1) by $\mathbf{v}_n$, (2) by $\mathbf{v}_i$, (4) by $\mathbf{B}$ and integrate in $\Omega$. The following terms vanish: for $j = n, i$,

$$\int_\Omega \mathbf{v}_j \cdot \nabla \mathbf{v}_j \cdot \mathbf{v}_j \, \mathrm{d}V = \frac{1}{2} \int_\Omega \mathbf{v}_j \cdot \nabla v_j^2 \, \mathrm{d}V = \frac{1}{2} \int_{\partial\Omega} v_j^2 \mathbf{v}_j \cdot \mathbf{n} \, \mathrm{d}\sigma = 0 \tag{7}$$

$$\int_\Omega \nabla P_j \cdot \mathbf{v}_j \, \mathrm{d}V = \int_{\partial\Omega} P_j \mathbf{v}_j \cdot \mathbf{n} \, \mathrm{d}\sigma = 0. \tag{8}$$

On the other hand, $\mathbf{J} \times \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{B} - (1/2)\nabla B^2$,

$$\int_\Omega \nabla B^2 \cdot \mathbf{v}_i \, \mathrm{d}V = \int_{\partial\Omega} B^2 \mathbf{v}_i \cdot \mathbf{n} \, \mathrm{d}\sigma = 0. \tag{9}$$

Also, $\nabla \times (\mathbf{v}_i \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v}_i - \mathbf{v}_i \cdot \nabla \mathbf{B}$, and

$$\int_\Omega (\mathbf{v}_i \cdot \nabla \mathbf{B}) \cdot \mathbf{B} \, \mathrm{d}V = \frac{1}{2} \int_{\partial\Omega} B^2 \mathbf{v}_i \cdot \mathbf{n} \, \mathrm{d}\sigma = 0. \tag{10}$$

Since $\mathbf{B} \times \mathbf{n} = \mathbf{0}$, $[(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}] \cdot \mathbf{n} = (\mathbf{J} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{n}) = 0$ at $\partial\Omega$, which implies

$$\int_\Omega [\nabla \times (\mathbf{J} \times \mathbf{B})] \cdot \mathbf{B} \, \mathrm{d}V = \int_{\partial\Omega} [(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}] \cdot \mathbf{n} \, \mathrm{d}\sigma + \int_\Omega (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{J} \, \mathrm{d}V = 0. \tag{11}$$

Finally

$$-\int_\Omega (\nabla \times \mathbf{J}) \cdot \mathbf{B} \, \mathrm{d}V = -\int_{\partial\Omega} (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{n} \, \mathrm{d}\sigma - \int_\Omega J^2 \, \mathrm{d}V = -\int_\Omega J^2 \, \mathrm{d}V. \tag{12}$$

The following identities result:

$$\frac{1}{2} \rho_n \frac{\partial}{\partial t} \int_\Omega v_n^2 \, \mathrm{d}V = k \int_\Omega \mathbf{v}_n \cdot (\mathbf{v}_i - \mathbf{v}_n) \, \mathrm{d}V, \tag{13}$$

$$\frac{1}{2}\rho_i \frac{\partial}{\partial t}\int_\Omega v_i^2 \, \mathrm{d}V = k\int_\Omega \mathbf{v}_i \cdot (\mathbf{v}_n - \mathbf{v}_i)\, \mathrm{d}V + \int_\Omega (\mathbf{B}\cdot\nabla\mathbf{B})\cdot\mathbf{v}_i \, \mathrm{d}V, \tag{14}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega B^2 \, \mathrm{d}V = -\eta\int_\Omega J^2\, \mathrm{d}V + \int_\Omega (\mathbf{B}\cdot\nabla\mathbf{v}_i)\cdot\mathbf{B}\, \mathrm{d}V. \tag{15}$$

Since $\mathbf{B}\cdot\nabla(\mathbf{v}_i\cdot\mathbf{B})$ integrates to zero, by adding these three identities we find

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \left(\rho_n v_n^2 + \rho_i v_i^2 + B^2\right) \mathrm{d}V = -k\int_\Omega |\mathbf{v}_i - \mathbf{v}_n|^2\, \mathrm{d}V - \eta\int_\Omega J^2\, \mathrm{d}V. \tag{16}$$

Hence the total (kinetic plus magnetic) energy $E(t)$ decreases in time. Since we have assumed $\eta \ll 1$, the main cause of this decrease is the friction between ions and neutrals, which ceases when both velocities coincide. Unless this is the case, the total energy would collapse catastrophically when $k\to\infty$.

From (16) and the positivity of the energy it follows

$$\int_0^\infty\int_\Omega |\mathbf{v}_i - \mathbf{v}_n|^2\, \mathrm{d}V\, \mathrm{d}t = \int_0^\infty \|\mathbf{v}_i - \mathbf{v}_n\|_2^2\, \mathrm{d}t < \infty, \tag{17}$$

$$\int_0^\infty \mathrm{d}t \int_\Omega J^2\, \mathrm{d}V < \infty. \tag{18}$$

Note that (18) does not hold in the presence of current sheets: our description is only valid as long as the MHD approximation holds. The integrability in time of $\|\mathbf{v}_i - \mathbf{v}_n\|_2^2$ is the key factor in the following arguments. A positive integrable function does not need to tend to zero when $t\to\infty$, although it really tends to zero for *most* times: since

$$\int_T^\infty \|\mathbf{v}_i - \mathbf{v}_n\|_2^2\, \mathrm{d}t \to 0$$

when $T\to\infty$ and this expression decreases with $T$, for every $\epsilon > 0$ there exists the smallest possible $T_\epsilon$ such that

$$\int_{T_\epsilon}^\infty \|\mathbf{v}_i - \mathbf{v}_n\|_2^2\, \mathrm{d}t \leqslant \epsilon^2.$$

Let $F_\epsilon \subset [T_\epsilon, \infty)$ be the set of instants $t$ such that $\|\mathbf{v}_i(t) - \mathbf{v}_n(t)\|_2^2 > \epsilon$. If we denote the one-dimensional measure by $m$,

$$\epsilon m(F_\epsilon) \leqslant \int_{F_\epsilon} \|\mathbf{v}_i - \mathbf{v}_n\|_2^2\, \mathrm{d}t \leqslant \int_0^\infty \|\mathbf{v}_i - \mathbf{v}_n\|_2^2\, \mathrm{d}t \leqslant \epsilon^2,$$

we find $m(F_\epsilon)\leqslant\epsilon$. Thus, for all times $T\notin F_\epsilon$,

$$\|\mathbf{v}_i(T) - \mathbf{v}_n(T)\|_2^2 \leqslant \epsilon, \tag{19}$$

and this occurs for all $T$ outside a set of measure less than $\epsilon$. From now on, we will say that an interval $[T_1, T_2]$ is $\epsilon$-regular if $T_1 \geqslant T_\epsilon$, and neither of the $T_i$ belongs to $F_\epsilon$.

## 3. Functional spaces

We will recall here some basic facts about the function spaces that will be used later. $L^2(\Omega)$ is the space of square integrable functions in $\Omega$, whose norm we have denoted by $\|\;\|_2$. The Sobolev space $H^m(\Omega)$ is formed by the functions $m$ times differentiable, and whose differentials up to the order $m$ belong to $L^2(\Omega)$. The norm in $H^m(\Omega)$ is, except by equivalences,

$$\|f\|_{H^m} = \sum_{|\alpha|\leqslant m} \|D^\alpha f\|_2.$$

For $s > 0$ a real non-integer number, $H^s(\Omega)$ may be defined by interpolation. It happens that, in dimension three, $H^s(\Omega) \subset \mathcal{C}(\bar{\Omega})$ if $s > 3/2$, where $\bar{\Omega}$ represents the closure of $\Omega$, and also the maximum norm $\|f\|_\infty \leqslant M\|f\|_{H^s}$; in the future all universal constants, depending only on $\Omega$, will be denoted by $M$ to avoid proliferation of constants. We will assume that $\Omega$ is simply connected, i.e. every closed loop is continuously contractible to a point; this implies that every irrotational field is a gradient.

Let $X_0$ be the subspace of $L^2(\Omega)^3$ formed by the functions of divergence zero and whose normal component vanishes at $\partial\Omega$ (both operations in the sense of distributions). $X_0$ is a closed subspace of $L^2(\Omega)^3$, the orthogonal of $\nabla H^1(\Omega)$ (the gradients of functions of $H^1(\Omega)$). This orthogonality is easy and instructive to prove in one direction:

$$\int_\Omega \mathbf{f}\cdot\nabla\psi\,\mathrm{d}V = \int_\Omega \nabla\cdot(\psi\mathbf{f})\,\mathrm{d}V = \int_{\partial\Omega} \psi\mathbf{f}\cdot\mathbf{n}\,\mathrm{d}\sigma = 0.$$

Hence

$$L^2(\Omega)^3 = X_0 \oplus \nabla H^1(\Omega).$$

This decomposition may be generalized: if $X_s = H^s(\Omega)^3 \cap X_0$,

$$H^s(\Omega)^3 = X_s \oplus \nabla H^{s+1}(\Omega). \tag{20}$$

This decomposition is not orthogonal with respect to the inner product of the space $H^s$, but nonetheless it represents a topological sum, in the sense that both projections are continuous. These projections form a Hodge-like decomposition of the functions of $H^s(\Omega)$, and may be found as follows: we set $\mathbf{f} = \mathbf{g}+\nabla\psi$. Since we demand that $\nabla\cdot\mathbf{g} = 0$, necessarily $\nabla\cdot\mathbf{f} = \Delta\psi$. The condition $\mathbf{g}\cdot\mathbf{n} = 0$ at $\partial\Omega$ completes the definition of $\psi$ as the solution of the Neumann problem

$$\Delta\psi = \nabla\cdot\mathbf{f}, \qquad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = \mathbf{f}\cdot\mathbf{n}.$$

The solution is unique except for an additive constant, which will not matter as we are only interested in $\nabla\psi$. Standard results on elliptic problems tell us that if we fix the solution of the Neumann problem by some other condition such as $\int_\Omega \psi\,\mathrm{d}V = 0$, we have

$$\|\psi\|_{H^{s+1}} \leqslant M\|\nabla\cdot\mathbf{f}\|_{H^{s-1}};$$

therefore

$$\|\nabla\psi\|_{H^s} \leqslant M\|\mathbf{f}\|_{H^s},$$

which proves the continuity of the second projection in (20); since both projections add to the identity, the first one is also continuous.

The same argument may be applied to the Hölder spaces $\mathcal{C}^{k,\alpha}(\Omega)$, $0 < \alpha < 1$, formed by the functions $k$ times differentiable whose $k$th differentials satisfy a Hölder condition of order $\alpha$. In particular

$$\mathcal{C}^{1,\alpha}(\Omega)^3 = (\mathcal{C}^{1,\alpha}(\Omega)^3 \cap X_0) \oplus \nabla(\mathcal{C}^{2,\alpha}(\Omega)). \tag{21}$$

Let us consider now a different subspace $Y_s$ of $H^s(\Omega)^3$: the functions of divergence zero such that its *tangential* component vanishes at the boundary. It is known that the curl operator takes $Y_s$ bijectively in $X_{s-1}$, and in fact the mapping is bicontinuous with the respective $H^s$ norms (see e.g. [9] for the case $s = 1$; the generalization is straightforward). This implies that if we denote by $H^s_t(\Omega)$ the subspace of the functions of $H^s(\Omega)^3$ formed by the functions whose tangential component vanishes at the boundary, the curl operator $\nabla\times : H^{s+1}_t(\Omega) \to X_s$ is surjective and continuous. In fact $H^{s+1}_t(\Omega)$ is the topological sum of $Y_{s+1}$ and the kernel of the curl, formed by the gradients of functions constant at the boundary.

Finally recall that the subspace $H^s_0(\Omega)$ is formed by the functions vanishing at the boundary, and its dual space is denoted by $H^{-s}(\Omega)$. We will handle a series of inequalities in dual norms (weak estimates), all of which will be stronger than $H^{-s}$ estimates.

## 4. Averages in $\epsilon$-regular intervals

By subtracting (2) from (1), we find

$$\frac{\partial(\mathbf{v}_n-\mathbf{v}_i)}{\partial t}=\mathbf{v}_i\cdot\nabla\mathbf{v}_i-\mathbf{v}_n\cdot\nabla\mathbf{v}_n-\frac{1}{\rho_n}\nabla P_n+\frac{1}{\rho_i}\nabla P_i +k\left(\frac{1}{\rho_n}+\frac{1}{\rho_i}\right)(\mathbf{v}_i-\mathbf{v}_n)-\frac{1}{\rho_i}(\mathbf{J}\times\mathbf{B}). \tag{22}$$

Let $\mathbf{f}\in X_s, s>5/2$. (Analogous results may be found by assuming $\mathbf{f}\in\mathcal{C}^{1,\alpha}(\Omega)\cap X_0$). Multiplying (22) by $\mathbf{f}$ and integrating $\Omega$, we find

$$\int_\Omega(\mathbf{J}\times\mathbf{B})\cdot\mathbf{f}\,\mathrm{d}V=\rho_i\frac{\partial}{\partial t}\int_\Omega(\mathbf{v}_i-\mathbf{v}_n)\cdot\mathbf{f}\,\mathrm{d}V+\rho_i\int_\Omega(\mathbf{v}_n\cdot\nabla\mathbf{f}\cdot\mathbf{v}_n-\mathbf{v}_i\cdot\nabla\mathbf{f}\cdot\mathbf{v}_i)\,\mathrm{d}V +k\left(\frac{\rho_i}{\rho_n}+1\right)\int_\Omega(\mathbf{v}_i-\mathbf{v}_n)\cdot\mathbf{f}\,\mathrm{d}V, \tag{23}$$

where we have used the identities

$$\int_\Omega\mathbf{v}_i\cdot\nabla(\mathbf{v}_i\cdot\mathbf{f})\,\mathrm{d}V=\int_\Omega\mathbf{v}_n\cdot\nabla(\mathbf{v}_n\cdot\mathbf{f})\mathrm{d}V=0\qquad\int_\Omega\nabla P_i\cdot\mathbf{f}\,\mathrm{d}V=\int_\Omega\nabla P_n\cdot\mathbf{f}\,\mathrm{d}V=0.$$

We have

$$\left|\int_\Omega\mathbf{v}_i\cdot\nabla\mathbf{f}\cdot\mathbf{v}_i-\mathbf{v}_n\cdot\nabla\mathbf{f}\cdot\mathbf{v}_n\,\mathrm{d}V\right|=\left|\int_\Omega\mathbf{v}_n\cdot\nabla\mathbf{f}\cdot(\mathbf{v}_i-\mathbf{v}_n)+(\mathbf{v}_i-\mathbf{v}_n)\cdot\nabla\mathbf{f}\cdot\mathbf{v}_i\,\mathrm{d}V\right| \leqslant(\|\mathbf{v}_n\|_2+\|\mathbf{v}_i\|_2)\|\nabla\mathbf{f}\|_\infty\|\mathbf{v}_i-\mathbf{v}_n\|_2\leqslant 2E(0)\|\mathbf{f}\|_{H^s}\|\mathbf{v}_i-\mathbf{v}_n\|_2. \tag{24}$$

$E(0)$ represents the total energy at the instant zero. Also

$$\left|\int_\Omega(\mathbf{v}_i-\mathbf{v}_n)\cdot\mathbf{f}\,\mathrm{d}V\right|\leqslant\|\mathbf{v}_i-\mathbf{v}_n\|_2\|\mathbf{f}\|_2\leqslant\|\mathbf{v}_i-\mathbf{v}_n\|_2\|\mathbf{f}\|_{H^s}. \tag{25}$$

Hence

$$\pm\int_\Omega(\mathbf{J}\times\mathbf{B})\cdot\mathbf{f}\,\mathrm{d}V\leqslant\pm\rho_i\frac{\partial}{\partial t}\int_\Omega(\mathbf{v}_i-\mathbf{v}_n)\cdot\mathbf{f}\,\mathrm{d}V +k\left(\frac{\rho_i}{\rho_n}+1\right)\|\mathbf{v}_i-\mathbf{v}_n\|_2\|\mathbf{f}\|_{H^s}+2\rho_iE(0)\|\mathbf{v}_i-\mathbf{v}_n\|_2\|\mathbf{f}\|_{H^s}. \tag{26}$$

Integrating in the interval $[T_1,T_2]$ yields

$$\left|\int_{T_1}^{T_2}(\mathbf{J}\times\mathbf{B},\mathbf{f})\,\mathrm{d}t\right|\leqslant\rho_i\big|\|(\mathbf{v}_i-\mathbf{v}_n)(T_2)\|_2-\|(\mathbf{v}_i-\mathbf{v}_n)(T_1)\|_2\big|\|\mathbf{f}\|_{H^s} +\left[k\left(\frac{\rho_i}{\rho_n}+1\right)+2\rho_iE(0)\right]\|\mathbf{f}\|_{H^s}\int_{T_1}^{T_2}\|\mathbf{v}_i-\mathbf{v}_n\|_2\,\mathrm{d}t. \tag{27}$$

The inequality of Schwarz yields

$$\int_{T_1}^{T_2}\|\mathbf{v}_i-\mathbf{v}_n\|_2\,\mathrm{d}t\leqslant\sqrt{T_2-T_1}\left(\int_{T_1}^{T_2}\|\mathbf{v}_i-\mathbf{v}_n\|_2^2\,\mathrm{d}t\right)^{1/2}\leqslant\epsilon\sqrt{T_2-T_1}.$$

Since we assume that $T_1,T_2\notin F_\epsilon$,

$$\big|\|(\mathbf{v}_i-\mathbf{v}_n)(T_2)\|_2-\|(\mathbf{v}_i-\mathbf{v}_n)(T_1)\|_2\big|\leqslant\epsilon,$$

which proves that the time average

$$\langle\mathbf{J}\times\mathbf{B}\rangle_{T_1,T_2}=\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\mathbf{J}\times\mathbf{B}\,\mathrm{d}t,$$

satisfies the following inequality:

$$\left|\left(\langle \mathbf{J}\times\mathbf{B}\rangle_{T_1,T_2},\mathbf{f}\right)\right| \leqslant \frac{1}{\sqrt{T_2-T_1}}\left(\frac{2\rho_i}{\sqrt{T_2-T_1}}+k\left(\frac{\rho_i}{\rho_n}+1\right)+2\rho_i E(0)\right)\right)\epsilon\|\mathbf{f}\|_{H^s}. \tag{28}$$

This represents a weak bound on $\langle \mathbf{J}\times\mathbf{B}\rangle$. We see that it cannot be used to obtain an estimate on $\mathbf{J}\times\mathbf{B}$ at a precise instant, because the right-hand side of the inequality goes to infinity: the larger the interval and the smaller the $\epsilon$ (i.e. for large times) the sharper is the estimate. Unfortunately, the space $X_s$ is very particular and one does not get an idea of the real size of $\langle \mathbf{J}\times\mathbf{B}\rangle$ only in terms of its action on $X_s$. To solve this, we recall the Hodge decomposition on (20):

$$H^s(\Omega)^3 = X_s \oplus \nabla H^{s+1}(\Omega).$$

The dual space of $X_s$ may be identified with the dual of $H^s(\Omega)^3$ modulo the functionals vanishing at $X_s$. Those have the form

$$\mathbf{f}\to\int_\Omega \mathbf{f}\cdot\nabla\psi\,\mathrm{d}V,$$

for some $\psi$ such that $\nabla\psi\in(H^s(\Omega)^3)^*$: thus $\psi$ is a distribution of order at most one. By including all the constants (including those corresponding to the projections in the topological sum) in a single $M$, we find that there exists $\psi$ such that

$$\left\|\langle \mathbf{J}\times\mathbf{B}\rangle_{T_1,T_2}-\nabla\psi\right\|_{H^s(\Omega)^*} \leqslant \frac{M}{\sqrt{T_2-T_1}}\left(1+\frac{1}{\sqrt{T_2-T_1}}\right)\epsilon. \tag{29}$$

Thus the averages of $\mathbf{J}\times\mathbf{B}$ are close in the $(H^s)^*$ weak sense to gradients for most of the intervals $[T_1,T_2]$ (i.e. when neither of $T_i$ belongs to $F_\epsilon$). This is an approximation of the equation for magnetostatic equilibria $\mathbf{J}\times\mathbf{B}=\nabla p$. The fact that solutions of this equation are notoriously difficult to find and are usually unstable, or at best marginally stable, proves that for most times the averages of the magnetic field lie near a narrow set of possible states. In [7, 12], it is asserted that the current must be concentrated in sheets while the rest of the plasma is current free: we are unable to reach this conclusion in our more general setting.

It seems logical that since the Lorentz force is close to a gradient, its curl (and therefore the Hall term) must be small at least in time average. We can prove this as follows: by the surjective and continuous character of the curl operator when considered from $H_t^{s+1}(\Omega)$ into $X_s$ as stated before, we may rewrite (28) as

$$\left|\left(\langle \mathbf{J}\times\mathbf{B}\rangle_{T_1,T_2},\nabla\times\mathbf{F}\right)\right| \leqslant \frac{M}{\sqrt{T_2-T_1}}\left(1+\frac{1}{\sqrt{T_2-T_1}}\right)\epsilon\|\mathbf{F}\|_{H^{s+1}}. \tag{30}$$

Since $[(\mathbf{J}\times\mathbf{B})\times\mathbf{F}]\cdot\mathbf{n}=0$ at $\partial\Omega$ for all $\mathbf{F}\in H_t^{s+1}(\Omega)$, (30) means

$$\left|\left(\langle \nabla\times(\mathbf{J}\times\mathbf{B})\rangle_{T_1,T_2},\mathbf{F}\right)\right| \leqslant \frac{M}{\sqrt{T_2-T_1}}\left(1+\frac{1}{\sqrt{T_2-T_1}}\right)\epsilon\|\mathbf{F}\|_{H^{s+1}}, \tag{31}$$

i.e.

$$\left\|\langle \nabla\times(\mathbf{J}\times\mathbf{B})\rangle_{T_1,T_2}\right\|_{H_t^{s+1}(\Omega)^*} \leqslant \frac{M}{\sqrt{T_2-T_1}}\left(1+\frac{1}{\sqrt{T_2-T_1}}\right)\epsilon. \tag{32}$$

Note that since $H_0^{s+1}(\Omega)^3\subset H_t^{s+1}(\Omega)$, the dual norm of the last space is stronger than the $H^{-s-1}(\Omega)$ norm. Thus the Hall term has in average a small size for $\epsilon$-regular intervals.

Note that so far our conclusions do not depend on the size of $h$. In fact the Hall term in the induction equation has played no role at all, since it disappears in the energy identities. Thus the fact that $\|\mathbf{v}_i-\mathbf{v}_n\|_2$ is square integrable in time, and its consequence concerning the proximity of the Lorentz force to a gradient, are independent of the Hall effect. Hence the rarity of episodes of kinetic energy transfer is general in the presence of ambipolar diffusion, but this does not mean that the Hall effect does not play any role in the number of such events and in the behaviour of the field in them. After all, the Hall term occurs in the induction equation, which governs the global evolution. It is only that energy inequalities, which form the basis of our arguments, are blind to the Hall term. Interestingly, a model where independently of ion-neutral diffusion, rare events of fast magnetic reconnection occur between large intervals of slow (Sweet–Parker) one, exists [10]; and fast reconnection due only to ambipolar diffusion has also been proposed [11]. Thus both mechanisms are plausible, and it is likely that both play a role in our situation; but the integral inequalities we handle are unable to discriminate between them.

Let us now study the behaviour of both kinetic energies and the magnetic one at the $\epsilon$-regular intervals. Since (13) implies

$$\rho_n \frac{\partial}{\partial t}\|\mathbf{v}_n\|_2^2 = k\int_\Omega (\mathbf{v}_i - \mathbf{v}_n)\cdot\mathbf{v}_n \,\mathrm{d}V \leqslant k\|\mathbf{v}_i - \mathbf{v}_n\|_2\|\mathbf{v}_n\|_2 \leqslant kE(0)\|\mathbf{v}_i - \mathbf{v}_n\|_2, \tag{33}$$

integrating in any time interval $[T_1, T_2]$,

$$\rho_n\left|\|\mathbf{v}_n(T_2)\|_2^2 - \|\mathbf{v}_n(T_1)\|_2^2\right| \leqslant kE(0)\sqrt{T_2 - T_1}\left(\int_{T_1}^{T_2}\|\mathbf{v}_i - \mathbf{v}_n\|_2^2\,\mathrm{d}t\right)^{1/2}. \tag{34}$$

Hence, for all $T_1 \geqslant T_\epsilon$,

$$\left|\|\mathbf{v}_n(T_2)\|_2^2 - \|\mathbf{v}_n(T_1)\|_2^2\right| \leqslant \frac{kE(0)}{\rho_n}\epsilon\sqrt{T_2 - T_1}. \tag{35}$$

This estimate is independent of the set $F_\epsilon$. It says that

$$\left|\|\mathbf{v}_n(T_2)\|_2^2 - \|\mathbf{v}_n(T_1)\|_2^2\right| = o(\sqrt{T_2 - T_1}),$$

when $T_1 \to \infty$; it does not imply that $\|\mathbf{v}_n\|_2$ tends to become constant, since any growth of the form $\|\mathbf{v}_n\|_2 \sim t^\beta$, $\beta < (1/4)$, would satisfy the bound. Still, (35) does not allow very rapid variations of the neutrals' kinetic energy.

For the rest of the energies we need to assume that the interval $[T_1, T_2]$ is $\epsilon$-regular. In this case, (19) together with (35) yields

$$\begin{aligned}\left|\|\mathbf{v}_i(T_2)\|_2^2 - \|\mathbf{v}_i(T_1)\|_2^2\right| &\leqslant |\|\mathbf{v}_i(T_2)\|_2 - \|\mathbf{v}_n(T_2)\|_2| \cdot |\|\mathbf{v}_i(T_2)\|_2 + \|\mathbf{v}_n(T_2)\|_2| \\ &\quad + \left|\|\mathbf{v}_n(T_2)\|_2^2 - \|\mathbf{v}_n(T_1)\|_2^2\right| + \left|\|\mathbf{v}_i(T_1)\|_2 - \|\mathbf{v}_n(T_1)\|_2\right| \cdot |\|\mathbf{v}_i(T_1)\|_2 + \|\mathbf{v}_n(T_1)\|_2| \\ &\leqslant 4\sqrt{\epsilon}\sqrt{E(0)} + \frac{kE(0)}{\rho_n}\epsilon\sqrt{T_2 - T_1}.\end{aligned} \tag{36}$$

Finally, since the total energy satisfies (16), we find

$$\begin{aligned}\left|\|\mathbf{B}(T_2)\|_2^2 - \|\mathbf{B}(T_1)\|_2^2\right| &\leqslant \rho_n\left|\|\mathbf{v}_n(T_2)\|_2^2 - \|\mathbf{v}_n(T_1)\|_2^2\right| + \rho_i\left|\|\mathbf{v}_i(T_2)\|_2^2 - \|\mathbf{v}_i(T_1)\|_2^2\right| \\ &\quad + k\int_{T_1}^{T_2}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2\,\mathrm{d}t + \eta\int_{T_1}^{T_2}\|\mathbf{J}\|_2^2\,\mathrm{d}t,\end{aligned} \tag{37}$$

which, together with (35), (36) and the integrability of the $\|\mathbf{v}_n - \mathbf{v}_i\|_2^2$ and $\|\mathbf{J}\|_2^2$, yields a similar bound for the difference of magnetic energies in the limit of an $\epsilon$-regular interval.

## 5. Behaviour of the magnetic field during transfers of kinetic energy

We are interested in the intervals where there is a rather large jump in the value of $\|\mathbf{v}_n - \mathbf{v}_i\|_2^2$ between the points $T_1$ and $T_2$. Thus, let us take

$$\left|\|\mathbf{v}_i(T_2) - \mathbf{v}_n(T_2)\|_2^2 - \|\mathbf{v}_i(T_1) - \mathbf{v}_n(T_1)\|_2^2\right| \geqslant R. \tag{38}$$

We will see that there are two possible causes for this jump: either the flow has large gradients at any of the $T_j$, or the Lorentz force and the Hall term are large there. While both are intuitive, the second cause is the most likely one for large times, because we will see that the gradient of $\mathbf{v}_n$ must be much larger than the Lorentz force to produce the same effect. Therefore we can reasonably identify these jumps with episodes of magnetic reconnection, which will transfer magnetic energy to $\mathbf{v}_i$ suddenly and therefore increase its difference with $\mathbf{v}_n$. We start again from (22), but this time we multiply it by $\mathbf{v}_n - \mathbf{v}_i$:

$$\frac{\partial}{\partial t}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2 = \mathbf{v}_i \cdot \nabla\mathbf{v}_i \cdot (\mathbf{v}_n - \mathbf{v}_i) - \mathbf{v}_n \cdot \nabla\mathbf{v}_n \cdot (\mathbf{v}_n - \mathbf{v}_i)$$
$$- k\left(\frac{1}{\rho_n} + \frac{1}{\rho_i}\right)\|\mathbf{v}_i - \mathbf{v}_i\|_2^2 - \frac{1}{\rho_n}\nabla P_n \cdot (\mathbf{v}_n - \mathbf{v}_i)$$
$$+ \frac{1}{\rho_i}\nabla P_i \cdot (\mathbf{v}_n - \mathbf{v}_i) + \frac{1}{\rho_i}(\mathbf{J}\times\mathbf{B}) \cdot (\mathbf{v}_n - \mathbf{v}_i). \quad (39)$$

Therefore

$$\frac{\partial}{\partial t}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2 = \mathbf{v}_i \cdot \nabla(\mathbf{v}_i - \mathbf{v}_n) \cdot (\mathbf{v}_n - \mathbf{v}_i) + (\mathbf{v}_i - \mathbf{v}_n) \cdot \nabla\mathbf{v}_n \cdot (\mathbf{v}_n - \mathbf{v}_i)$$
$$- k\left(\frac{1}{\rho_i} + \frac{1}{\rho_n}\right)\|\mathbf{v}_n - \mathbf{v}_i\|_2^2 - \frac{1}{\rho_n}\nabla P_n \cdot (\mathbf{v}_n - \mathbf{v}_i)$$
$$+ \frac{1}{\rho_n}\nabla P_i \cdot (\mathbf{v}_n - \mathbf{v}_i) - \frac{1}{\rho_i}(\mathbf{J}\times\mathbf{B}) \cdot (\mathbf{v}_n - \mathbf{v}_i). \quad (40)$$

Integrating in $\Omega$, the term $\mathbf{v}_i \cdot \nabla(|\mathbf{v}_i - \mathbf{v}_n|^2)$ and the ones involving the pressures disappear and we are left with

$$\frac{\partial}{\partial t}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2 = -\int_\Omega (\mathbf{v}_n - \mathbf{v}_i) \cdot \nabla\mathbf{v}_n \cdot (\mathbf{v}_n - \mathbf{v}_i)\,\mathrm{d}V$$
$$- k\left(\frac{1}{\rho_n} + \frac{1}{\rho_i}\right)\int_\Omega \|\mathbf{v}_n - \mathbf{v}_i\|_2^2\,\mathrm{d}V - \frac{1}{\rho_i}\int_\Omega (\mathbf{J}\times\mathbf{B}) \cdot (\mathbf{v}_n - \mathbf{v}_i)\,\mathrm{d}V. \quad (41)$$

Integrating this identity in time and using some elementary inequalities,

$$\left|\|\mathbf{v}_n(T_2) - \mathbf{v}_i(T_2)\|_2^2 - \|\mathbf{v}_n(T_1) - \mathbf{v}_i(T_1)\|_2^2\right| \leqslant \sup_{[T_1,T_2]}\|\nabla\mathbf{v}_n\|_\infty \int_{T_1}^{T_2}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2\,\mathrm{d}t$$
$$+ k\left(\frac{1}{\rho_n} + \frac{1}{\rho_i}\right)\int_{T_1}^{T_2}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2\,\mathrm{d}t + \left|\int_{T_1}^{T_2}\mathrm{d}t\int_\Omega (\mathbf{J}\times\mathbf{B}) \cdot (\mathbf{v}_n - \mathbf{v}_i)\,\mathrm{d}V\right|. \quad (42)$$

Let $P : L^2(\Omega) \to X_0$ denote the orthogonal projection. Then

$$\int_\Omega (\mathbf{J}\times\mathbf{B}) \cdot (\mathbf{v}_n - \mathbf{v}_i)\mathrm{d}V = \int_\Omega P(\mathbf{J}\times\mathbf{B}) \cdot (\mathbf{v}_n - \mathbf{v}_i)\,\mathrm{d}V.$$

Let

$$C_0 = \sup_{[T_1,T_2]}\|\nabla\mathbf{v}_n\|_\infty + k\left(\frac{1}{\rho_n} + \frac{1}{\rho_i}\right).$$

Assume as stated that (38) holds. By using Schwarz's inequality,

$$R \leqslant C_0\int_{T_1}^{T_2}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2\,\mathrm{d}t + \left(\int_\Omega \|P(\mathbf{J}\times\mathbf{B})\|_2^2\,\mathrm{d}t\right)^{1/2}\left(\int_{T_1}^{T_2}\|\mathbf{v}_n - \mathbf{v}_i\|_2^2\,\mathrm{d}t\right)^{1/2}. \quad (43)$$

Since for $T_1$ large the integral of $\|\mathbf{v}_n - \mathbf{v}_i\|_2^2$ is very small, the greatest contribution to the right-hand side comes from the norm of the projection of the Lorentz force. It is true that a large $C_0$ (either because of the velocity gradient or because of $k$) would increase the influence of the purely kinetic term of (43) and thus decrease the importance of the Lorentz force. Still, for large times the integral of $\|\mathbf{v}_n - \mathbf{v}_i\|_2^2$ is much smaller than its square root, so in the long run it is the second term which predominates. Of course $\|P(\mathbf{J} \times \mathbf{B})\|_2 \leqslant \|\mathbf{J} \times \mathbf{B}\|_2$, but the projected term is more meaningful, since it represents the rotational part of the Lorentz force. Recall that if we endow $X_0$ with the $L^2$-norm, $Y_1$ with the $H^1$-norm, the curl operator is an isomorphism between both spaces. Hence the adjoint operator is an isomorphism between their duals; this adjoint operator coincides formally with $\nabla\times$ in the sense of distributions. If we consider the scalar product in $L^2$ as the dual action, $X_0$ is its own dual and $\nabla\times : X_0 \to Y_1^*$ is an isomorphism. In particular,

$$\|P(\mathbf{J} \times \mathbf{B})\|_2 \leqslant M\|\nabla \times P(\mathbf{J} \times \mathbf{B})\|_{Y_1^*} = M\|\nabla \times (\mathbf{J} \times \mathbf{B})\|_{Y_1^*}. \tag{44}$$

This, together with (43), proves that the weak norm $(H_t^1)^*$ (and therefore the $H^{-1}$-norm) of the Hall term must be large at the jump episodes. Equation (43) shows too that the more advanced the time, the larger must be the Lorentz force to obtain sizeable bursts in $\|\mathbf{v}_n - \mathbf{v}_i\|_2$; thus in the long run the possible bursts tend to disappear and both plasma species will flow together. As explained in the previous section, the magnetic field will tend to be near configurations of magnetostatic equilibria. This, of course, could be prevented by the presence of non-gradient forcing terms in either of the momentum equations or in the induction one. Forcing terms are usually present in astrophysical models, as well as compressibility of the plasma, so that our results cannot be said to predict in all cases a quasi stable evolution punctuated by progressively smaller events where there is energy interchange in all real physical situations. Still, this would be the case would the flow be left to its own devices.

Let us recall that what we have proved rigorously is the intermittency of reconnection events, in the presence or the absence of the Hall effect; and that at these events, both the Lorentz force and the Hall term are large, which intuitively seem to point in the direction of magnetic reconnection. Unfortunately, we cannot prove that these events occur at all. Their possible existence for large times could contradict the proposed enhancement of turbulent diffusion by ambipolar drift, [7, 12] agreeing more with [13], which shows that a turbulent neutral flow and strong magnetic field reduce turbulent diffusion.

## 6. Conclusions

When two species of fluids, a ionized and a neutral one, coexist with a magnetic field the magnetohydrodynamic evolution of the flow is conditioned by the drag due to the collisions between both species. In fact the difference of velocities is square integrable, which only allows for progressively shorter intervals where both flows differ markedly from one another. It is proved that in the quiescent periods, where this difference is small, the Lorentz force is close in average to a gradient, thus satisfying approximately the equation of magnetostatic equilibria. Also its curl, the Hall term, is small; the kinetic energies of the ionized and the neutral flows, as well as the magnetic energy, vary slowly, which means that there are no rapid transfers from one type of energy to the other. The opposite is true in the intervals where the difference of both velocities is large: the Lorentz force (and therefore the current) and the Hall term are extremely large, in particular for advanced times, which accounts for the ingredients of fast magnetic reconnection. Eventually the bursts tend to die out, unless kept by the presence of some forcing.

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